REE From EOF

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Abstract

It is well-known that entanglement of formation (EOF) and relative entropy of entanglement (REE) are exactly identical for all two-qubit pure states even though their definitions are completely different. We think this fact implies that there is a veiled connection between EOF and REE. In this context, we suggest a procedure, which enables us to compute REE from EOF without relying on the converse procedure. It is shown that the procedure yields correct REE for many symmetric mixed states such as Bell-diagonal, generalized Vedral-Plenino, and generalized Horodecki states. It also gives a correct REE for less symmetric Vedral-Plenio-type state. However, it is shown that the procedure does not provide correct REE for arbitrary mixed states.

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I. INTRODUCTION

Entanglement of formation (EOF)[1] and relative entropy of entanglement (REE)[2, 3] are two major entanglement monotones for bipartite systems. For pure states $\rho = |\psi\rangle\langle\psi|$ the EOF $\mathcal{E}_F(\rho)$ is defined as a von Neumann entropy of its subsystem $\rho_A = \mathrm{tr}_B \rho$. On the contrary, REE is defined as minimum value of the relative entropy with separable states;

$$\mathcal{E}_R(\rho) = \min_{\sigma \in \mathcal{D}} \mathrm{tr}(\rho \ln \rho - \rho \ln \sigma), \tag{1.1}$$

where $\mathcal{D}$ is a set of separable states[1]. It was shown in Ref.[3] that $E_R(\rho)$ is a upper bound of the distillable entanglement[1]. The separable state σ_*, which yields a minimum value of the relative entropy is called the closest separable state (CSS) of ρ. Surprising fact, at least for us, is that although definitions of EOF and REE are completely different, they are exactly same for all pure states[3]. This fact may indicate that they are related to each other although the exact connection is not revealed yet. The main purpose of this paper is to explore the veiled connection between EOF and REE.

For mixed states ρ EOF is defined via a convex-roof method[1, 5];

$$\mathcal{E}_F(\rho) = \min \sum_i p_i \mathcal{E}_F(\rho_i), \tag{1.2}$$

where the minimum is taken over all possible pure-state decompositions with $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. The ensemble that gives the minimum value in Eq.(1.2) is called the optimal decomposition of the mixed state ρ. Thus, the main task for analytic calculation of EOF is derivation of an optimal decomposition of the given mixture. Few years ago, the procedure for construction of the optimal decomposition was derived[6, 7] in the two-qubit system, the simplest bipartite system, by making use of the time-reversal operation of spin-1/2 particles appropriately. In these references the relation

$$\mathcal{E}_F(C) = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \tag{1.3}$$

is used, where $h(x)$ is a binary entropy function $h(x) = -x \ln x - (1-x) \ln(1-x)$ and C is called the concurrence. This procedure, usually called Wootters procedure, was re-examined

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[1] Since REE is defined through another separable state σ, it is called "distance entanglement measure". Another example of the distance entanglement measure is a geometric entanglement measure defined as $E_g(\psi) = 1 - P_{max}$, where P_{max} is a maximal overlap of a given state $|\psi\rangle$ with the nearest product state[4].

in Ref.[5] in terms of antilinearity. Introduction of antilinearity in quantum information theory makes it possible to derive concurrence-based entanglement monotones for tripartite[8] and multipartite systems[9]. Due to the discovery of the closed formula for EOF in the two-qubit system, EOF is recently applied not only to quantum information theory but also to many scientific fields such as life science[10].

While EOF is used in various areas of science, REE is not because of its calculational difficulty. In order to obtain REE analytically for given mixed state ρ one should derive its CSS, but still we don't know how to derive CSS[11] even in the two-qubit system except very rare cases[3, 13, 14]. In Ref.[13] REE for Bell-diagonal, generalized Vedral-Plenio, and generalized Horodecki states were derived analytically through pure geometric arguments[12].

Due to the notorious difficulty some people try to solve the REE problem conversely. Let σ_* be a two-qubit boundary states in the convex set of the separable states. In Ref.[15] authors derived entangled states, whose CSS are σ_*. This converse procedure is extended to the qudit system[16] and is generalized as convex optimization problems[17]. However, as emphasized in Ref.[13] still it is difficult to find a CSS σ_* of given entangled state ρ although the converse procedure may provide some useful information on the CSS[14].

In this paper we will try to find a CSS for given entangled two-qubit state without relying on the converse procedure. As commented, EOF and REE are identical for bipartite pure states although they are defined differently. This means that they are somehow related to each other. If this connection is unveiled, probably we can find CSS for arbitrary two-qubit mixed states because we already know how to compute EOF through Wootters procedure. To explore this issue is original motivation of this paper. We will show in the following that REE of many mixed symmetric states can be analytically obtained from EOF if one follows the following procedure:

(1) For entangled two-qubit state ρ let $\rho = \sum_j p_j \rho_j$ ($\rho_j = |\psi_j\rangle\langle\psi_j|$) be an optimal decomposition for calculation of EOF.

(2) Since ρ_j are pure states, it is possible to obtain their CSS σ_j. Thus, it is straight to derive a separable mixture $\tilde{\sigma} = \sum_j p_j \sigma_j$.

(3) If $\tilde{\sigma}$ is a boundary state in the convex set of separable states, the procedure is terminated with $\sigma_* = \tilde{\sigma}$.

(4) If $\tilde{\sigma}$ is not a boundary state, we consider $\pi = q\rho + (1-q)\tilde{\sigma}$. By requiring that π is a boundary state, one can fix q, *say* $q = q_0$. Then we identify $\sigma_* = q_0\rho + (1-q_0)\tilde{\sigma}$.

This procedure is schematically represented in Fig. 1.

The paper is organized as follows. In section II we prove again that EOF and REE are identical for all pure states, because the results of this section are used in next sections. In section III we show that the procedure generates the correct CSS for Bell-diagonal states. In section IV and section V we show that the procedure generates the correct CSS for generalized Vedral-Plenio and generalized Horodecki states, respectively. In section VI we consider two less symmetric states, Vedral-Plenio-type and Horodecki-type states. It is shown that while the procedure generates a correct CSS for the former, it does not give a correct one for the latter. In section VII a brief conclusion is given.
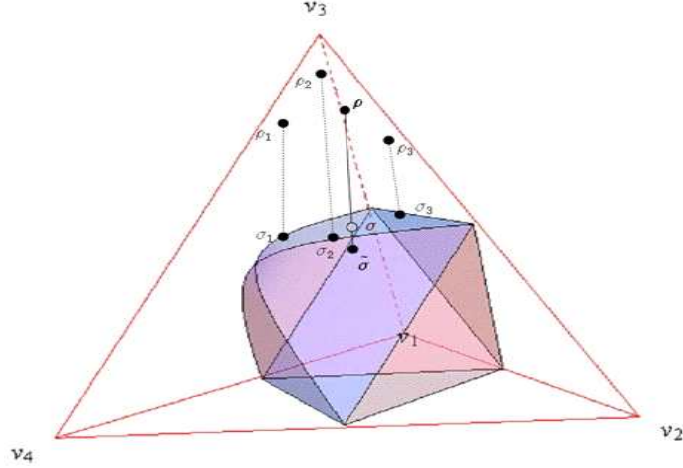

FIG. 1: (Color online) The schematic diagram of the procedure, by which REE can be computed from EOF. The polygon at the center is a deformed octahedron[12, 13]. Inside and outside of the octahedron separable and entangled states reside, respectively. The CSS of the entangled state resides at the surface of the octahedron.

II.   REE AND EOF FOR PURE STATES

In this section we will show that REE and EOF are identical for two-qubit pure states. This fact was already proven in Theorem 3 of Ref.[3]. We will prove this again more directly,

because explicit Schmidt bases are used in next sections.

Let us consider a general two-qubit pure state $|\psi_2\rangle_{AB} = \alpha_1|00\rangle + \alpha_2|01\rangle + \alpha_3|10\rangle + \alpha_4|11\rangle$ with $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1$. Then, its concurrence is $\mathcal{C} = 2|\alpha_1\alpha_4 - \alpha_2\alpha_3|$. Now, we define

$$x_\pm = \frac{\alpha_1^*\alpha_2 + \alpha_3^*\alpha_4}{\mathcal{N}_\pm} \qquad y_\pm = \frac{\lambda_\pm - (|\alpha_1|^2 + |\alpha_3|^2)}{\mathcal{N}_\pm} \tag{2.1}$$

where

$$\lambda_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \mathcal{C}^2}\right] \qquad \mathcal{N}_\pm^2 = |\alpha_1^*\alpha_2 + \alpha_3^*\alpha_4|^2 + |\lambda_\pm - (|\alpha_1|^2 + |\alpha_3|^2)|^2. \tag{2.2}$$

Now, we consider 2×2 matrix u, whose components u_{ij} are

$$
\begin{aligned}
u_{11} &= \alpha_1\left(\frac{|x_+|^2}{\sqrt{\lambda_+}} + \frac{|x_-|^2}{\sqrt{\lambda_-}}\right) + \alpha_2\left(\frac{x_+^*y_+}{\sqrt{\lambda_+}} + \frac{x_-^*y_-}{\sqrt{\lambda_-}}\right) \\
u_{12} &= \alpha_1\left(\frac{x_+y_+^*}{\sqrt{\lambda_+}} + \frac{x_-y_-^*}{\sqrt{\lambda_-}}\right) + \alpha_2\left(\frac{|y_+|^2}{\sqrt{\lambda_+}} + \frac{|y_-|^2}{\sqrt{\lambda_-}}\right) \\
u_{21} &= \alpha_3\left(\frac{|x_+|^2}{\sqrt{\lambda_+}} + \frac{|x_-|^2}{\sqrt{\lambda_-}}\right) + \alpha_4\left(\frac{x_+^*y_+}{\sqrt{\lambda_+}} + \frac{x_-^*y_-}{\sqrt{\lambda_-}}\right) \\
u_{22} &= \alpha_3\left(\frac{x_+y_+^*}{\sqrt{\lambda_+}} + \frac{x_-y_-^*}{\sqrt{\lambda_-}}\right) + \alpha_4\left(\frac{|y_+|^2}{\sqrt{\lambda_+}} + \frac{|y_-|^2}{\sqrt{\lambda_-}}\right).
\end{aligned}
\tag{2.3}
$$

Then Schmidt bases for each party are defined as

$$|i_A\rangle = \sum_{j=0}^{1} v_{ji}|j\rangle \qquad |i_B\rangle = \sum_{k=0}^{1} w_{ik}|k\rangle \quad (i = 0, 1) \tag{2.4}$$

where

$$v = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}\begin{pmatrix} x_+ & x_- \\ y_+ & y_- \end{pmatrix} \qquad w = \begin{pmatrix} x_+^* & y_+^* \\ x_-^* & y_-^* \end{pmatrix}. \tag{2.5}$$

Using Eq. (2.4), it is easy to show that $|\psi_2\rangle_{AB}$ and its CSS σ_* are

$$|\psi_2\rangle_{AB} = \sqrt{\lambda_+}|0_A0_B\rangle + \sqrt{\lambda_-}|1_A1_B\rangle \tag{2.6}$$

$$\sigma_* = \lambda_+|0_A0_B\rangle\langle 0_A0_B| + \lambda_-|1_A1_B\rangle\langle 1_A1_B|.$$

Applying Eq. (1.1), one can show easily $\mathcal{E}_R(|\psi_2\rangle) = -\lambda_+ \ln\lambda_+ - \lambda_- \ln\lambda_-$, which is exactly the same with EOF.

III. BELL-DIAGONAL STATES

In this section we will show that the procedure mentioned above solves the REE problem of the Bell-diagonal states:

$$\rho_{BD} = \lambda_1|\beta_1\rangle\langle\beta_1| + \lambda_2|\beta_2\rangle\langle\beta_2| + \lambda_3|\beta_3\rangle\langle\beta_3| + \lambda_4|\beta_4\rangle\langle\beta_4| \qquad (3.1)$$

where $\sum_{j=1}^{4}\lambda_j = 1$, and

$$|\beta_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\beta_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad (3.2)$$

$$|\beta_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\beta_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

The CSS and REE of ρ_{BD} were obtained in many literatures[3, 13, 14] through various different methods. If, for convenience, $\max(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_3$, the CSS and REE of ρ_{BD} are

$$\pi_{BD} = \frac{\lambda_1}{2(1-\lambda_3)}|\beta_1\rangle\langle\beta_1| + \frac{\lambda_2}{2(1-\lambda_3)}|\beta_2\rangle\langle\beta_2| + \frac{1}{2}|\beta_3\rangle\langle\beta_3| + \frac{\lambda_4}{2(1-\lambda_3)}|\beta_4\rangle\langle\beta_4| \quad (3.3)$$

$$\mathcal{E}_R(\rho_{BD}) = -h(\lambda_3) + \ln 2.$$

Now, we will show that the procedure we suggested also yields the same result. Following Wootters procedure, one can show that the optimal decomposition of ρ_{BD} for $\lambda_3 \geq 1/2$[2] is

$$\rho_{BD} = \sum_{j=0}^{4} p_j|\psi_j^{BD}\rangle\langle\psi_j^{BD}| \qquad (3.4)$$

where $p_j = 1/4 \quad (j = 1, \cdots, 4)$ and

$$|\psi_1^{BD}\rangle = \sqrt{\lambda_1}|\beta_1\rangle + i\sqrt{\lambda_2}|\beta_2\rangle + \sqrt{\lambda_3}|\beta_3\rangle + \sqrt{\lambda_4}|\beta_4\rangle$$

$$|\psi_2^{BD}\rangle = \sqrt{\lambda_1}|\beta_1\rangle + i\sqrt{\lambda_2}|\beta_2\rangle - \sqrt{\lambda_3}|\beta_3\rangle - \sqrt{\lambda_4}|\beta_4\rangle \qquad (3.5)$$

$$|\psi_3^{BD}\rangle = \sqrt{\lambda_1}|\beta_1\rangle - i\sqrt{\lambda_2}|\beta_2\rangle + \sqrt{\lambda_3}|\beta_3\rangle - \sqrt{\lambda_4}|\beta_4\rangle$$

$$|\psi_4^{BD}\rangle = \sqrt{\lambda_1}|\beta_1\rangle - i\sqrt{\lambda_2}|\beta_2\rangle - \sqrt{\lambda_3}|\beta_3\rangle + \sqrt{\lambda_4}|\beta_4\rangle.$$

All $|\psi_j^{BD}\rangle \quad (j = 1, \cdots, 4)$ have the same concurrence $\mathcal{C} = 2\lambda_3 - 1$ and, hence, the same $\lambda_{\pm}$ (defined in Eq. (2.2)) as

$$\lambda_{\pm} = \frac{1}{2}\left(\sqrt{\lambda_3} \pm \sqrt{1-\lambda_3}\right)^2. \qquad (3.6)$$

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[2] If $\lambda_3 \leq 1/2$, ρ_{BD} is a separable state.

The Schmidt bases of $|\psi_1^{BD}\rangle$ can be explicitly derived by following the procedure of the previous section and the result is

$$
\begin{aligned}
|0_A\rangle &= \frac{1}{N_+}\left[\left(\sqrt{1-\lambda_3}+\sqrt{\lambda_4}\right)|0\rangle+\left(\sqrt{\lambda_1}-i\sqrt{\lambda_2}\right)|1\rangle\right]\\
|1_A\rangle &= \frac{-1}{N_-}\left[\left(\sqrt{1-\lambda_3}-\sqrt{\lambda_4}\right)|0\rangle-\left(\sqrt{\lambda_1}-i\sqrt{\lambda_2}\right)|1\rangle\right]\\
|0_B\rangle &= \frac{1}{N_+}\left[\left(\sqrt{\lambda_1}+i\sqrt{\lambda_2}\right)|0\rangle+\left(\sqrt{1-\lambda_3}+\sqrt{\lambda_4}\right)|1\rangle\right]\\
|1_B\rangle &= \frac{1}{N_-}\left[\left(\sqrt{\lambda_1}+i\sqrt{\lambda_2}\right)|0\rangle-\left(\sqrt{1-\lambda_3}-\sqrt{\lambda_4}\right)|1\rangle\right],
\end{aligned}
\tag{3.7}
$$

where the normalization constants $N_\pm$ are

$$
N_\pm = \sqrt{2\sqrt{1-\lambda_3}\left(\sqrt{1-\lambda_3}\pm\sqrt{\lambda_4}\right)}.
\tag{3.8}
$$

Thus the CSS of $|\psi_1^{BD}\rangle$, say σ_1, can be straightforwardly computed by making use of Eq. (2.6);

$$
\begin{aligned}
\sigma_1 &= \lambda_+|0_A0_B\rangle\langle0_A0_B|+\lambda_-|1_A1_B\rangle\langle1_A1_B|\\
&= \frac{1}{4\left(1-\lambda_3\right)}\begin{pmatrix}
\mu\mu^* & \mu\nu_+ & \mu\nu_- & \mu^2\\
\mu^*\nu_+ & d_+ & \mu\mu^* & \mu\nu_+\\
\mu^*\nu_- & \mu\mu^* & d_- & \mu\nu_-\\
(\mu^*)^2 & \mu^*\nu_+ & \mu^*\nu_- & \mu\mu^*
\end{pmatrix},
\end{aligned}
\tag{3.9}
$$

where

$$
\mu = \sqrt{\lambda_1}+i\sqrt{\lambda_2} \qquad \nu_\pm = 2\left(1-\lambda_3\right)\sqrt{\lambda_3}\pm\sqrt{\lambda_4}
\tag{3.10}
$$
$$
d_\pm = \left(1-\lambda_3+\lambda_4\right)\pm4\left(1-\lambda_3\right)\sqrt{\lambda_3\lambda_4}.
$$

Similarly, one can derive the Schmidt bases for other $|\psi_j^{BD}\rangle$ $(j=2,3,4)$ and the corresponding CSS σ_j. Then, one can show that the separable state $\tilde\sigma=\sum_{j=1}^{4}p_j\sigma_j$ with $p_j=1/4$ for all j is

$$
\tilde\sigma = \frac{1}{4\left(1-\lambda_3\right)}\begin{pmatrix}
\lambda_1+\lambda_2 & 0 & 0 & \lambda_1-\lambda_2\\
0 & 1-\lambda_3+\lambda_4 & \lambda_1+\lambda_2 & 0\\
0 & \lambda_1+\lambda_2 & 1-\lambda_3+\lambda_4 & 0\\
\lambda_1-\lambda_2 & 0 & 0 & \lambda_1+\lambda_2
\end{pmatrix}.
\tag{3.11}
$$

Obviously, This is a boundary state in the convex set of the separable states, because the minimal eigenvalue of its partial transposition, *say* $\tilde\sigma^\Gamma$, is zero. Thus, the procedure

mentioned in the Introduction is terminated with identifying $\sigma_* = \tilde{\sigma}$. In fact, it is easy to show that $\tilde{\sigma}$ is exactly the same with π_{BD} in Eq. (3.3). Thus, the procedure we suggested correctly derives the CSS of the Bell-diagonal states.

IV. GENERALIZED VEDRAL-PLENIO STATE

In this section we will derive the CSS of the generalized Vedral-Plenio (GVP) state defined as

$$\rho_{vp} = \lambda_1 |\beta_3\rangle\langle\beta_3| + \lambda_2 |01\rangle\langle01| + \lambda_3 |10\rangle\langle10| \qquad (\lambda_1 + \lambda_2 + \lambda_3 = 1) \tag{4.1}$$

by following the procedure mentioned above. In fact the CSS and REE of the GVP were explicitly derived in Ref.[13] using a geometric argument, which are

$$\pi_{vp} = \left(\frac{\lambda_1}{2} + \lambda_2\right) |01\rangle\langle01| + \left(\frac{\lambda_1}{2} + \lambda_3\right) |10\rangle\langle10| \tag{4.2}$$

$$\mathcal{E}_R(\rho_{vp}) = H\left(\frac{\lambda_1}{2} + \lambda_2\right) - H(\Lambda_+)$$

where

$$\Lambda_\pm = \frac{1}{2}\left[1 \pm \sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2}\right]. \tag{4.3}$$

Now, we define

$$a = \frac{\lambda_1 \Lambda_+}{\sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2}} \qquad b = -\frac{(\lambda_2 - \lambda_3)\sqrt{\Lambda_+ \Lambda_-}}{\sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2}} \qquad c = -\frac{\lambda_1 \Lambda_-}{\sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2}} \tag{4.4}$$

and $\Omega^2 = 2[(a-c)^2 + 4b^2 - (a-c)\sqrt{(a-c)^2 + 4b^2}]$. We also define the unnormalized states $|v_\pm\rangle = \sqrt{\Lambda_\pm}|\Lambda_\pm\rangle$, where $|\Lambda_\pm\rangle$ are eigenstates of ρ_{vp};

$$|\Lambda_+\rangle = \frac{1}{N}\left[\left(\sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2} + (\lambda_2 - \lambda_3)\right)|01\rangle + \lambda_1|10\rangle\right] \tag{4.5}$$

$$|\Lambda_-\rangle = \frac{1}{N}\left[\lambda_1|01\rangle - \left(\sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2} + (\lambda_2 - \lambda_3)\right)|10\rangle\right].$$

In Eq. (4.5) N is a normalization constant given by

$$N^2 = 2\sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2}\left\{\sqrt{\lambda_1^2 + (\lambda_2 - \lambda_3)^2} + (\lambda_2 - \lambda_3)\right\}. \tag{4.6}$$

Then, following Ref.[7], the optimal decomposition of ρ_{vp} for EOF is $\rho_{vp} = \sum_{j=1}^{2} p_j |\psi_j^{VP}\rangle\langle\psi_j^{VP}|$, where $p_1 = p_2 = 1/2$ and

$$|\psi_1^{VP}\rangle = \frac{-i}{\Omega}\left[2b - i\left\{\sqrt{(a-c)^2 + 4b^2} - (a-c)\right\}\right](|v_+\rangle + i|v_-\rangle) \tag{4.7}$$

$$|\psi_2^{VP}\rangle = \frac{-i}{\Omega}\left[2b + i\left\{\sqrt{(a-c)^2 + 4b^2} - (a-c)\right\}\right](|v_+\rangle - i|v_-\rangle).$$

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Following section II one can derive the CSS for $|\psi_j^{VP}\rangle$ directly. Then, one can realize that $|\psi_1^{VP}\rangle$ and $|\psi_2^{VP}\rangle$ have the same CSS, which is identical with π_{vp}. Thus, the procedure also gives a correct CSS for the GVP states.

V.   GENERALIZED HORODECKI STATES

In this section we will show that the procedure also generates the correct CSS for the generalized Horodecki states

$$\rho_H = \lambda_1|\beta_3\rangle\langle\beta_3|+\lambda_2|00\rangle\langle00|+\lambda_3|11\rangle\langle11| \tag{5.1}$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1 \geq 2\sqrt{\lambda_2\lambda_3}{}^3$. The CSS and REE of ρ_H were derived in Ref.[13] using a geometrical argument and the results are

$$\pi_H = \frac{(\lambda_1 + 2\lambda_2)(\lambda_1 + 2\lambda_3)}{2}|\beta_3\rangle\langle\beta_3|+\frac{(\lambda_1 + 2\lambda_2)^2}{4}|00\rangle\langle00|+\frac{(\lambda_1 + 2\lambda_3)^2}{4}|11\rangle\langle11|$$

$$\mathcal{E}_R(\rho_H) = \lambda_1 \ln \lambda_1 + \lambda_2 \ln \lambda_2 + \lambda_3 \ln \lambda_3 + 2H\left(\frac{\lambda_1}{2} + \lambda_2\right) - \lambda_1 \ln 2. \tag{5.2}$$

Following Ref.[7] one can straightforwardly construct the optimal decomposition of ρ_H for EOF, which is $\rho_H = \sum_{j=1}^3 p_j|\psi_j^H\rangle\langle\psi_j^H|$, where $p_1 = p_2 = p_3 = 1/3$ and

$$\begin{aligned}
|\psi_1^H\rangle &= \sqrt{\lambda_1}|\beta_3\rangle + \sqrt{\lambda_2}|00\rangle + \sqrt{\lambda_3}|11\rangle \\
|\psi_2^H\rangle &= \sqrt{\lambda_1}|\beta_3\rangle + \sqrt{\lambda_2}e^{i2\pi/3}|00\rangle + \sqrt{\lambda_3}e^{-i2\pi/3}|11\rangle \\
|\psi_3^H\rangle &= \sqrt{\lambda_1}|\beta_3\rangle + \sqrt{\lambda_2}e^{i4\pi/3}|00\rangle + \sqrt{\lambda_3}e^{-i4\pi/3}|11\rangle.
\end{aligned} \tag{5.3}$$

In order to treat $|\psi_j^H\rangle$ as an unified manner let us consider $|\phi\rangle = \sqrt{\lambda_1}|\beta_3\rangle + \sqrt{\lambda_2}e^{i\theta}|00\rangle + \sqrt{\lambda_3}e^{-i\theta}|11\rangle$. Then, $\lambda_\pm$ defined in Eq. (2.2) is

$$\lambda_\pm = \left(\frac{R \pm \left(\sqrt{\lambda_2} + \sqrt{\lambda_3}\right)}{2}\right)^2 \tag{5.4}$$

where $R = \sqrt{2\lambda_1 + \left(\sqrt{\lambda_2} - \sqrt{\lambda_3}\right)^2}$. Since $\lambda_\pm$ is independent of θ, this fact indicates that $\lambda_\pm$ of $|\psi_j^H\rangle$ are equal to Eq.(5.4) for all j. Following section II, it is straightforward to show

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^{3} If $\lambda_1 \leq 2\sqrt{\lambda_2\lambda_3}$, ρ_H becomes a separable state.

that the Schmidt bases of $|\phi\rangle$ are

$$|0_A\rangle = \sqrt{\frac{\lambda_1}{R\left[R - \left(\sqrt{\lambda_2} - \sqrt{\lambda_3}\right)\right]}}|0\rangle + \sqrt{\frac{R - \left(\sqrt{\lambda_2} - \sqrt{\lambda_3}\right)}{2R}}e^{-i\theta}|1\rangle$$

$$|1_A\rangle = -\sqrt{\frac{\lambda_1}{R\left[R + \left(\sqrt{\lambda_2} - \sqrt{\lambda_3}\right)\right]}}|0\rangle + \sqrt{\frac{R + \left(\sqrt{\lambda_2} - \sqrt{\lambda_3}\right)}{2R}}e^{-i\theta}|1\rangle \qquad (5.5)$$

$$|0_B\rangle = e^{i\theta}|0_A\rangle \qquad |1_B\rangle = -e^{i\theta}|1_A\rangle.$$

Then the CSS σ_ϕ of $|\phi\rangle$ is

$$
\begin{aligned}
\sigma_\phi &\equiv \lambda_+|0_A 0_B\rangle\langle 0_A 0_B| + \lambda_-|1_A 1_B\rangle\langle 1_A 1_B| \\
&= \begin{pmatrix}
\frac{\lambda_1 + 2\lambda_2}{2} - \frac{\lambda_1}{2R^2} & \mathcal{A}e^{i\theta} & \mathcal{A}e^{i\theta} & \frac{\lambda_1}{2R^2}e^{2i\theta} \\
\mathcal{A}e^{-i\theta} & \frac{\lambda_1}{2R^2} & \frac{\lambda_1}{2R^2} & \mathcal{B}e^{i\theta} \\
\mathcal{A}e^{-i\theta} & \frac{\lambda_1}{2R^2} & \frac{\lambda_1}{2R^2} & \mathcal{B}e^{i\theta} \\
\frac{\lambda_1}{2R^2}e^{-2i\theta} & \mathcal{B}e^{-i\theta} & \mathcal{B}e^{-i\theta} & \frac{\lambda_1 + 2\lambda_3}{2} - \frac{\lambda_1}{2R^2}
\end{pmatrix}
\end{aligned} \qquad (5.6)
$$

where

$$\mathcal{A} = \frac{\sqrt{2\lambda_1}}{4R^2}\left[2\sqrt{\lambda_2} + \left(\sqrt{\lambda_2} + \sqrt{\lambda_3}\right)\left(\lambda_1 - 2\sqrt{\lambda_2\lambda_3}\right)\right] \qquad (5.7)$$

$$\mathcal{B} = \frac{\sqrt{2\lambda_1}}{4R^2}\left[2\sqrt{\lambda_3} + \left(\sqrt{\lambda_2} + \sqrt{\lambda_3}\right)\left(\lambda_1 - 2\sqrt{\lambda_2\lambda_3}\right)\right].$$

Thus, the CSS σ_j of $|\psi_j^H\rangle$ can be obtained by letting $\theta = 0, 2\pi/3, 4\pi/3$, respectively.

Then, $\tilde{\sigma} = \sum_{j=1}^{3} p_j \sigma_j$ with $p_j = 1/3$ $(j = 1, 2, 3)$ reduces

$$
\tilde{\sigma} = \begin{pmatrix}
\frac{\lambda_1 + 2\lambda_2}{2} - \frac{\lambda_1}{2R^2} & 0 & 0 & 0 \\
0 & \frac{\lambda_1}{2R^2} & \frac{\lambda_1}{2R^2} & 0 \\
0 & \frac{\lambda_1}{2R^2} & \frac{\lambda_1}{2R^2} & 0 \\
0 & 0 & 0 & \frac{\lambda_1 + 2\lambda_3}{2} - \frac{\lambda_1}{2R^2}
\end{pmatrix}. \qquad (5.8)
$$

However, $\tilde{\sigma}$ is not a boundary state in the convex set of the separable states, because the minimum eigenvalue of $\tilde{\sigma}^\Gamma$ is positive. Thus, we define

$$\sigma_* = x\tilde{\sigma} + (1-x)\rho_H. \qquad (0 \le x \le 1) \qquad (5.9)$$

The condition that the minimum eigenvalue of σ_*^Γ is zero fixes x as

$$x = \frac{R^2}{2\lambda_1}\left(\lambda_1 + 2\sqrt{\lambda_2\lambda_3}\right). \qquad (5.10)$$

Inserting Eq.(5.10) into σ_*, one can show that σ_* reduces to π_H. Thus, our procedure gives a correct CSS for the generalized Horodecki states.

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VI.   LESS SYMMETRIC STATES

In the previous sections we have shown that the procedure generates the correct CSS and REE for various symmetric states such as Bell-diagonal, GVP, and generalized Horodecki states. In this section we will apply the procedure to the less symmetric states.

A.   Vedral-Plenio-Type State

The first quantum state we consider is

$$\Sigma_1 = A_2|01\rangle\langle01| + A_3|10\rangle\langle10| + D\left(|01\rangle\langle10| + |10\rangle\langle01|\right), \tag{6.1}$$

where $A_2 + A_3 = 1$, $A_2 \geq A_3$ and $0 \leq D \leq \sqrt{A_2 A_3}$. Of course, if $A_2 = \frac{\lambda_1}{2} + \lambda_2$, $A_3 = \frac{\lambda_1}{2} + \lambda_3$, and $D = \frac{\lambda_1}{2}$, Σ_1 reduces to ρ_{vp} in Eq. (4.1). Thus, we call Σ_1 as Vedral-Plenio-type state.

In order to apply the procedure to Σ_1 we introduce

$$R = \sqrt{(A_2 - A_3)^2 + 4D^2} \qquad \tan 2\theta = \frac{2D}{A_2 - A_3}$$

$$\lambda_1 = \frac{1}{2}\left[(A_2 + A_3) + R\right] \qquad \lambda_2 = \frac{1}{2}\left[(A_2 + A_3) - R\right] \tag{6.2}$$

$$|\lambda_1\rangle = \cos\theta|01\rangle + \sin\theta|10\rangle \qquad |\lambda_2\rangle = \sin\theta|01\rangle - \cos\theta|10\rangle.$$

Applying Ref.[7], it is possible to derive the optimal decomposition of Σ_1 for EOF; $\Sigma_1 = p_1|w_1\rangle\langle w_1| + p_2|w_2\rangle\langle w_2|$, where

$$p_1 = \frac{1}{2}\left[1 + \frac{A_2 - A_3}{\sqrt{1 - 4D^2}}\right] \qquad p_2 = \frac{1}{2}\left[1 - \frac{A_2 - A_3}{\sqrt{1 - 4D^2}}\right] \tag{6.3}$$

and

$$|w_1\rangle = \frac{1}{\mathcal{Y}_+}\left[\left(\sqrt{\xi_+\eta_+} + \sqrt{\xi_-\eta_-}\right)\sqrt{\lambda_1}|\lambda_1\rangle + \left(\sqrt{\xi_+\eta_-} - \sqrt{\xi_-\eta_+}\right)\sqrt{\lambda_2}|\lambda_2\rangle\right] \tag{6.4}$$

$$|w_2\rangle = \frac{1}{\mathcal{Y}_-}\left[\left(\sqrt{\xi_+\eta_-} - \sqrt{\xi_-\eta_+}\right)\sqrt{\lambda_1}|\lambda_1\rangle - \left(\sqrt{\xi_+\eta_+} + \sqrt{\xi_-\eta_-}\right)\sqrt{\lambda_2}|\lambda_2\rangle\right].$$

In Eq. (6.4) $\xi_\pm$, $\eta_\pm$, and $\mathcal{Y}_\pm$ are

$$\xi_\pm = R\sqrt{A_2 A_3} \pm D(A_2 + A_3) \qquad \eta_\pm = \sqrt{A_2 A_3(1 - 4D^2)} \pm D(A_2 - A_3) \tag{6.5}$$

$$\mathcal{Y}_\pm^2 = 2A_2 A_3 R\left[\sqrt{1 - 4D^2} \pm (A_2 - A_3)\right].$$

Following section II, one can derive the CSS σ_1 and σ_2 of $|w_1\rangle$ and $|w_2\rangle$ after long and tedious calculation. The final results are

$$
\begin{aligned}
\sigma_1 =& \left[\frac{\cos\theta\sqrt{\lambda_1}\left(\sqrt{\xi_+}\eta_+ + \sqrt{\xi_-}\eta_-\right) + \sin\theta\sqrt{\lambda_2}\left(\sqrt{\xi_+}\eta_- - \sqrt{\xi_-}\eta_+\right)}{\mathcal{Y}_+} \right]^2 |01\rangle\langle01| \\
& + \left[\frac{\sin\theta\sqrt{\lambda_1}\left(\sqrt{\xi_+}\eta_+ + \sqrt{\xi_-}\eta_-\right) - \cos\theta\sqrt{\lambda_2}\left(\sqrt{\xi_+}\eta_- - \sqrt{\xi_-}\eta_+\right)}{\mathcal{Y}_+} \right]^2 |10\rangle\langle10| \quad (6.6) \\
\sigma_2 =& \left[\frac{\cos\theta\sqrt{\lambda_1}\left(\sqrt{\xi_+}\eta_- - \sqrt{\xi_-}\eta_+\right) - \sin\theta\sqrt{\lambda_2}\left(\sqrt{\xi_+}\eta_+ + \sqrt{\xi_-}\eta_-\right)}{\mathcal{Y}_-} \right]^2 |01\rangle\langle01| \\
& + \left[\frac{\sin\theta\sqrt{\lambda_1}\left(\sqrt{\xi_+}\eta_- - \sqrt{\xi_-}\eta_+\right) + \cos\theta\sqrt{\lambda_2}\left(\sqrt{\xi_+}\eta_+ + \sqrt{\xi_-}\eta_-\right)}{\mathcal{Y}_-} \right]^2 |10\rangle\langle10|.
\end{aligned}
$$

Then, $\tilde{\sigma} = p_1\sigma_1 + p_2\sigma_2$ simply reduces to

$$
\tilde{\sigma} = A_2|01\rangle\langle01| + A_3|10\rangle|10\rangle. \tag{6.7}
$$

This is manifestly boundary state in the convex set of separable states. Thus, the procedure states that $\tilde{\sigma}$ is a CSS of Σ_1. This is exactly the same with theorem 1 of Ref.[14].

B. Horodecki-Type State

The second less symmetric quantum state we consider is

$$
\Sigma_2 = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A & D & 0 \\ 0 & D & A & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} \tag{6.8}
$$

where $A_1 + A_4 + 2A = 1$ and $\sqrt{A_1 A_4} < D \le A$. If $A = D = \lambda_1/2$, $A_1 = \lambda_2$, and $A_4 = \lambda_3$, Σ_2 reduces to ρ_H in Eq. (5.1). Thus, we call Σ_2 as Horodecki-type state. Applying Ref.[7], one can derive the optimal decomposition of Σ_2 for EOF as $\Sigma_2 = \sum_{j=1}^{4} p_j|h_j\rangle$, where $p_j = 1/4$

for all j and

$$
\begin{aligned}
|h_1\rangle &= \sqrt{A+D}|\beta_3\rangle + \sqrt{A-D}|\beta_4\rangle + \sqrt{A_1}|00\rangle + \sqrt{A_4}|11\rangle \\
|h_2\rangle &= \sqrt{A+D}|\beta_3\rangle + \sqrt{A-D}|\beta_4\rangle - \sqrt{A_1}|00\rangle - \sqrt{A_4}|11\rangle \\
|h_3\rangle &= \sqrt{A+D}|\beta_3\rangle - \sqrt{A-D}|\beta_4\rangle + i\sqrt{A_1}|00\rangle - i\sqrt{A_4}|11\rangle \\
|h_4\rangle &= \sqrt{A+D}|\beta_3\rangle - \sqrt{A-D}|\beta_4\rangle - i\sqrt{A_1}|00\rangle + i\sqrt{A_4}|11\rangle.
\end{aligned}
\tag{6.9}
$$

In order to consider $|h_j\rangle$ $(j = 1, \cdots, 4)$ all together, we define

$$
\begin{aligned}
|\varphi_1\rangle &= \sqrt{A+D}|\beta_3\rangle + \sqrt{A-D}|\beta_4\rangle + e^{i\theta}\sqrt{A_1}|00\rangle + e^{-i\theta}\sqrt{A_4}|11\rangle \\
|\varphi_2\rangle &= \sqrt{A+D}|\beta_3\rangle - \sqrt{A-D}|\beta_4\rangle + e^{i\theta}\sqrt{A_1}|00\rangle + e^{-i\theta}\sqrt{A_4}|11\rangle.
\end{aligned}
\tag{6.10}
$$

For $|\varphi_1\rangle$ the Schmidt bases are

$$
\begin{aligned}
|0_A\rangle &= \frac{1}{2\mathcal{Z}_+}\Bigg[\sqrt{2}\left(\sqrt{A-D}\sqrt{1+\mathcal{C}} + \sqrt{A+D}\sqrt{1-\mathcal{C}}\right)|0\rangle \\
&\quad + e^{-i\theta}\left\{ \left(\sqrt{A_1}+\sqrt{A_4}\right)\sqrt{1+\mathcal{C}} - \left(\sqrt{A_1}-\sqrt{A_4}\right)\sqrt{1-\mathcal{C}}\right\}|1\rangle \Bigg] \\
|1_A\rangle &= \frac{1}{2\mathcal{Z}_-}\Bigg[\sqrt{2}\left(\sqrt{A-D}\sqrt{1+\mathcal{C}} - \sqrt{A+D}\sqrt{1-\mathcal{C}}\right)|0\rangle \\
&\quad + e^{-i\theta}\left\{ \left(\sqrt{A_1}+\sqrt{A_4}\right)\sqrt{1+\mathcal{C}} + \left(\sqrt{A_1}-\sqrt{A_4}\right)\sqrt{1-\mathcal{C}}\right\}|1\rangle \Bigg] \\
|0_B\rangle &= \frac{1}{2\mathcal{Z}_+}\Bigg[\sqrt{2}e^{i\theta}\left\{ \sqrt{A+D}\left(\sqrt{A_1}+\sqrt{A_4}\right) + \sqrt{A-D}\left(\sqrt{A_1}-\sqrt{A_4}\right)\right\}|0\rangle \\
&\quad + \left\{ -(A_1-A_4) + 2\sqrt{A^2-D^2} + \sqrt{1-\mathcal{C}^2}\right\}|1\rangle \Bigg] \\
|1_B\rangle &= \frac{1}{2\mathcal{Z}_-}\Bigg[\sqrt{2}e^{i\theta}\left\{ \sqrt{A+D}\left(\sqrt{A_1}+\sqrt{A_4}\right) + \sqrt{A-D}\left(\sqrt{A_1}-\sqrt{A_4}\right)\right\}|0\rangle \\
&\quad + \left\{ -(A_1-A_4) + 2\sqrt{A^2-D^2} - \sqrt{1-\mathcal{C}^2}\right\}|1\rangle \Bigg],
\end{aligned}
\tag{6.11}
$$

where $\mathcal{C} = 2\left(D - \sqrt{A_1 A_4}\right)$ and

$$
\mathcal{Z}_\pm^2 = \frac{1}{2}\sqrt{1-\mathcal{C}^2}\left[\sqrt{1-\mathcal{C}^2} \mp (A_1 - A_4) \pm 2\sqrt{A^2-D^2}\right].
\tag{6.12}
$$

Thus, the CSS $\sigma_1(\theta)$ of $|\varphi_1\rangle$ is

$$
\sigma_1(\theta) = \left(\frac{\sqrt{1+\mathcal{C}}+\sqrt{1-\mathcal{C}}}{2}\right)^2 |0_A 0_B\rangle\langle 0_A 0_B| + \left(\frac{\sqrt{1+\mathcal{C}}-\sqrt{1-\mathcal{C}}}{2}\right)^2 |1_A 1_B\rangle\langle 1_A 1_B|.
\tag{6.13}
$$

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Similarly, it is straightforward to derive the CSS $\sigma_2(\theta)$of $|\varphi_2\rangle$ Then, one can show

$$\tilde{\Pi} \equiv \frac{1}{4}\left[\sigma_1(0) + \sigma_1(\pi) + \sigma_2\left(\frac{\pi}{2}\right) + \sigma_2\left(-\frac{\pi}{2}\right)\right]$$

$$= \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a & d & 0 \\ 0 & d & a & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \quad (6.14)$$

where

$$a_1 = \frac{1}{4(1-\mathcal{C}^2)}\left[(1+\mathcal{C})\left(\sqrt{A_1}+\sqrt{A_4}\right)^2 + (1-\mathcal{C})\left(\sqrt{A_1}-\sqrt{A_4}\right)^2\right.$$

$$\left.+2(1-\mathcal{C}^2)\,(A_1-A_4)\right]$$

$$a_4 = \frac{1}{4(1-\mathcal{C}^2)}\left[(1+\mathcal{C})\left(\sqrt{A_1}+\sqrt{A_4}\right)^2 + (1-\mathcal{C})\left(\sqrt{A_1}-\sqrt{A_4}\right)^2\right.$$

$$\left.-2(1-\mathcal{C}^2)\,(A_1-A_4)\right] \quad (6.15)$$

$$a = \frac{1}{2(1-\mathcal{C}^2)}\left[(1+\mathcal{C})\,(A-D) + (1-\mathcal{C})\,(A+D)\right]$$

$$d = \frac{2A\sqrt{A_1 A_4} + D\,(A_1+A_4)}{1-\mathcal{C}^2}.$$

One can show that if $A = D = \lambda_1/2$, $A_1 = \lambda_2$, and $A_4 = \lambda_3$, $\tilde{\Pi}$ reduces to Eq. (5.8).

Since $\tilde{\Pi}$ is not a boundary state in the set of separable states, we define

$$\Pi_* = x\tilde{\Pi} + (1-x)\Sigma_2. \quad (6.16)$$

Then, the CSS condition of Π_* is

$$[x\,(a_1 - A_1) + A_1]\,[x\,(a_4 - A_4) + A_4] = [x(d - D) + D]^2. \quad (6.17)$$

In the Horodecki state limit Eq. (6.17) gives a solution (5.10). Using $a_1 - A_1 = a_4 - A_4 = -(a - A) = f/(1-\mathcal{C}^2)$ and $d - D = g/(1-\mathcal{C}^2)$ where

$$f = \mathcal{C}\,(D - A\mathcal{C}) \qquad g = \mathcal{C}\,(\mathcal{C}D - A)\,, \quad (6.18)$$

the solution of x, *say* $x = x_*$, can be obtained by solving the quadratic equation (6.17). Inserting $x = x_*$ in Eq. (6.16), one can compute Π_* explicitly, which is a candidate of CSS for Σ_2.

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The CSS of Σ_2 was derived in the theorem 2 of Ref.[14] by using the converse procedure introduced in Ref.[15]. The explicit form of the CSS is

$$\pi_{\Sigma_2} = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r & y & 0 \\ 0 & y & r & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix} \tag{6.19}$$

where

$$r_1 = \frac{1}{F} \left[2A_1(A_1 + A_2)(A_1 + A_2 + A_4) - D^2(A_1 - A_4) + \Delta \right] \tag{6.20}$$

$$r_4 = \frac{1}{F} \left[2A_4(A_2 + A_4)(A_1 + A_2 + A_4) + D^2(A_1 - A_4) + \Delta \right]$$

$$r = \frac{1}{F} \left[2(A_1 + A_2)(A_2 + A_4)(A_1 + A_2 + A_4) - D^2(A_1 + 2A_2 + A_4) - \Delta \right]$$

and $y = \sqrt{r_1 r_4}$. In Eq. (6.20) D and Δ are

$$F = 2(A_1 + A_2 + A_4 + D)(A_1 + A_2 + A_4 - D) \tag{6.21}$$

$$\Delta = D\sqrt{D^2(A_1 - A_4)^2 + 4A_1 A_4 (A_1 + A_2)(A_2 + A_4)}.$$

Our candidate $\Pi_*|_{x=x_*}$ does not coincide with the correct CSS π_{Σ_2}. Thus, the procedure does not give a correct REE for Σ_2, although it gives correct REE for Bell-diagonal, GVP, generalized Horodecki, and Vedral-Plenio-type states.

VII. CONCLUSION

In this paper we examine the possibility for deriving the closed formula for REE in two-qubit system without relying on the converse procedure discussed in Ref.[15–17]. Since REE and EOF are identical for all pure states in spite of their different definitions, we think they should have some connection somehow. In this context we suggest a procedure, where REE can be computed from EOF. The procedure gives correct REE for many symmetric states such as Bell-diagonal, GVP, and generalized Horodecki states. It also generates a correct REE for less symmetric states such as Σ_1. However, the procedure failed to produce a correct REE for the less symmetric states Σ_2. This means our procedure is still incomplete for deriving the closed formula of REE.

We think still the connection between EOF and REE is not fully revealed. If this connection is sufficiently understood in the future, probably the closed formula for REE can be derived. We hope to explore this issue in the future.

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