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On an n-dimensional mixed type additive and quadratic functional equation

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ABSTRACT

In this paper, we investigate the generalized Hyers–Ulam stability of the functional equation

$$\sum_{k_2,\ldots,k_n=0}^{1} f\left(x_1+\sum_{i=2}^{n}(-1)^{k_i}x_i\right)-2^{n-1}f(x_1)-2^{n-2}\sum_{i=2}^{n}(f(x_i)+f(-x_i))=0$$

for integer values of n such that $n \geqslant 2$, where f is a function from a normed space X to a Banach space Y. The solutions of the equation are called additive–quadratic mappings.

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1. Introduction

A classical question in the theory of functional equations is "when is it true that a function which approximately satisfies a functional equation must be somehow close to an exact solution of the equation?" Such a problem was formulated by Ulam in 1940 and is called a *stability problem* of the functional equation (see [23]). In the following year, Hyers [7] gave a partial solution of Ulam's problem for the case of approximate additive functions. Subsequently, during the last seven decades, Hyers' theorem was generalized by several mathematicians worldwide in the context of a large variety of functional equations originating from functional analysis, differential equations, analytic number theory and geometry (cf. [1–6,8–22]).

Throughout this paper, assuming that $n \geqslant 2$ is an integer, X is a normed space, and that Y is a Banach space, we consider the *n-dimensional mixed type additive and quadratic functional equation*

$$\sum_{k_2,\ldots,k_n=0}^{1} f\left(x_1+\sum_{i=2}^{n}(-1)^{k_i}x_i\right)-2^{n-1}f(x_1)-2^{n-2}\sum_{i=2}^{n}(f(x_i)+f(-x_i))=0, \tag{1.1}$$

whose solutions are called *quadratic–additive mappings*.

In this paper, we investigate a general stability problem for the n-dimensional mixed type additive and quadratic functional equation (1.1).

2. Generalized Hyers–Ulam stability of equation (1.1)

In this section, we prove the generalized Hyers–Ulam stability of the n-dimensional mixed type additive and quadratic functional equation (1.1), where $n \geqslant 2$ is some integer.

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Let (s,t) be a fixed element of $\{(1,1),(1,-1),(-1,-1)\}$ and let $\varphi : X^n \to [0,\infty)$ be a function satisfying the conditions:

$$\sum_{i=0}^{\infty} 4^{-si}\varphi(2^{si}x_1, 2^{si}x_2, \ldots, 2^{si}x_n) < \infty, \tag{2.1}$$

$$\sum_{i=0}^{\infty} 2^{-ti}\varphi(2^{ti}x_1, 2^{ti}x_2, \ldots, 2^{ti}x_n) < \infty \tag{2.2}$$

for all $x_1, x_2, \ldots, x_n \in X$. For convenience, we use the following abbreviations for a given mapping $f : X \to Y$:

$$\begin{aligned}
&Df(x_1,x_2,\ldots,x_n) := \sum_{k_2,\ldots,k_n=0}^{1} f\left(x_1 + \sum_{i=2}^{n}(-1)^{k_i}x_i\right) - 2^{n-1}f(x_1) - 2^{n-2}\sum_{i=2}^{n}(f(x_i)+f(-x_i)),\\
&J_m f(x) := \frac{1}{2}\left(4^{-sm}(f(2^{sm}x) + f(-2^{sm}x)) + 2^{-tm}(f(2^{tm}x) - f(-2^{tm}x))\right)\\
&Af(x,y) := f(x+y) - f(x) - f(y),\\
&Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y),\\
&f_e(x) := \frac{1}{2}(f(x)+f(-x)),\\
&f_o(x) := \frac{1}{2}(f(x)-f(-x))
\end{aligned}$$

for all $x, y, x_1, x_2, \ldots, x_n \in X$ and all integers $m \geqslant 0$.

From these notations, if $f(0) = 0$, we get

$$\begin{aligned}
J_m f(x) - J_{m+1}f(x) = {}& \frac{4^{\tau_{-s,m}}}{2^{n-1}}Df(2^{\tau_{s,m}}x, 2^{\tau_{s,m}}x, 0, \ldots, 0)s + \frac{4^{\tau_{-s,m}}}{2^{n-1}}Df(-2^{\tau_{s,m}}x, -2^{\tau_{s,m}}x, 0, \ldots, 0)s\\
&+ \frac{2^{\tau_{-t,m}}}{2^{n-1}}Df(2^{\tau_{t,m}}x, 2^{\tau_{t,m}}x, 0, \ldots, 0)t - \frac{2^{\tau_{-t,m}}}{2^{n-1}}Df(-2^{\tau_{t,m}}x, -2^{\tau_{t,m}}x, 0, \ldots, 0)t
\end{aligned} \tag{2.3}$$

for all $x \in X$, where $\tau_{j,m}$ is the integer defined by

$$\tau_{j,m} = j\left(m + \frac{1}{2}\right) - \frac{1}{2}$$

for $j \in \{-1, 1\}$.

If f is a solution of the functional equation $Df(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X$, then f is called a quadratic–additive mapping.

Lemma 2.1. *A mapping $f : X \to Y$ is a solution of* (1.1) *if and only if f_e is a quadratic mapping and f_o is an additive mapping.*

Proof. Let $f : X \to Y$ satisfy $Df(x_1, x_2, \ldots, x_n) = 0$. Since $f(0) = \frac{Df(0,0,\ldots,0)}{2^{n-1}} = 0$, we get

$$\begin{aligned}
Qf_e(x,y) &= \frac{Df_e(x,y,0,\ldots,0)}{2^{n-2}} = 0,\\
Af_o(x,y) &= \frac{Df_o\left(\frac{x+y}{2},\frac{x+y}{2},0,\ldots,0\right) - Df_o\left(\frac{x+y}{2},\frac{x-y}{2},0,\ldots,0\right)}{2^{n-2}} = 0
\end{aligned}$$

for all $x, y \in X$, i.e., f_e is a quadratic mapping and f_o is an additive mapping.

Conversely, assume that f_e is a quadratic mapping and f_o is an additive mapping. Then we get

$$\begin{aligned}
&Df(x_1,x_2,\ldots,x_n) = Df_e(x_1,x_2,\ldots,x_n) + Df_o(x_1,x_2,\ldots,x_n) = \sum_{k_3,\ldots,k_n=0}^{1} Qf_e\left(x_1, x_2 + \sum_{i=3}^{n}(-1)^{k_i}x_i\right)\\
&+ 2\sum_{k_4,\ldots,k_n=0}^{1} Qf_e\left(x_2, x_3 + \sum_{i=4}^{n}(-1)^{k_i}x_i\right) + \cdots + 2^{n-3}\sum_{k_n=0}^{1} Qf_e\left(x_{n-2}, x_{n-1} + \sum_{i=n}^{n}(-1)^{k_i}x_i\right) + 2^{n-2}Qf_e(x_{n-1},x_n)\\
&+ \sum_{k_2,\ldots,k_n=0}^{1} Af_o\left(x_1, \sum_{i=2}^{n}(-1)^{k_i}x_i\right) = 0
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in X$, i.e., f is a solution of (1.1). □

In the following theorems, we will investigate the generalized Hyers–Ulam stability problems of the functional equation (1.1).

Theorem 2.2. *Suppose $f : X \to Y$ is a mapping such that*

$$\|Df(x_1, x_2, \ldots, x_n)\| \leqslant \varphi(x_1, x_2, \ldots, x_n) \tag{2.4}$$

for all $x_1, x_2, \ldots, x_n \in X$ with $f(0) = 0$. Then there exists a quadratic–additive mapping $F : X \to Y$ such that

$$DF(x_1, x_2, \ldots, x_n) = 0$$

for all $x_1, x_2, \ldots, x_n \in X$ and

$$\|f(x) - F(x)\| \leqslant \sum_{i=0}^{\infty} \Phi_i(x) \tag{2.5}$$

for all $x \in X$, where Φ_i is the mapping defined by

$$\Phi_i(x) := \frac{4^{\tau_{-s,i}}}{2^{n-1}} \left(\varphi(2^{\tau_{s,i}}x, 2^{\tau_{s,i}}x, 0, \ldots, 0) + \varphi(-2^{\tau_{s,i}}x, -2^{\tau_{s,i}}x, 0, \ldots, 0)\right) + \frac{2^{\tau_{-t,i}}}{2^{n-1}} \left(\varphi(2^{\tau_{t,i}}x, 2^{\tau_{t,i}}x, 0, \ldots, 0) + \varphi(-2^{\tau_{t,i}}x, -2^{\tau_{t,i}}x, 0, \ldots, 0)\right).$$

Proof. It follows from (2.3) and (2.4) that

$$\|J_m f(x) - J_{m+m'} f(x)\| \leqslant \sum_{i=m}^{m+m'-1} \left\| \frac{4^{\tau_{-s,i}}}{2^{n-1}} Df(2^{\tau_{s,i}}x, 2^{\tau_{s,i}}x, 0, \ldots, 0)s + \frac{4^{\tau_{-s,i}}}{2^{n-1}} Df(-2^{\tau_{s,i}}x, -2^{\tau_{s,i}}x, 0, \ldots, 0)s + \frac{2^{\tau_{-t,i}}}{2^{n-1}} Df(2^{\tau_{t,i}}x, 2^{\tau_{t,i}}x, 0, \ldots, 0)t - \frac{2^{\tau_{-t,i}}}{2^{n-1}} Df(-2^{\tau_{t,i}}x, -2^{\tau_{t,i}}x, 0, \ldots, 0)t \right\| \leqslant \sum_{i=m}^{m+m'-1} \Phi_i(x) \tag{2.6}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $m, m' \in \mathbb{N}$.

From (2.1), (2.2) and (2.6), it follows that the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J_m f(x)\}$ converges in Y. Hence, we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{m \to \infty} J_m f(x)$$

for all $x \in X$. Moreover, by putting $m = 0$ and letting $m' \to \infty$ in (2.6), we get (2.5). From the definition of F, we easily have

$$DF(x_1, x_2, \ldots, x_n) = \lim_{i \to \infty} \frac{1}{2} \left(4^{-si} Df\left(2^{si}x_1, \ldots, 2^{si}x_n\right) + 4^{-si} Df\left(-2^{si}x_1, \ldots, -2^{si}x_n\right) + 2^{-ti} Df\left(2^{ti}x_1, \ldots, 2^{ti}x_n\right) - 2^{-ti} Df\left(-2^{ti}x_1, \ldots, -2^{ti}x_n\right)\right) = 0$$

for all $x_1, x_2, \ldots, x_n \in X$. □

Theorem 2.3. *Let $s = -1$ or $t = 1$. Suppose $f : X \to Y$ is a mapping satisfying* (2.4) *for all $x_1, x_2, \ldots, x_n \in X$ with $f(0) = 0$. Then there exists a unique quadratic–additive mapping F satisfying* (2.5) *for all $x \in X$.*

Proof. The statement of this theorem follows from Theorem 2.2 except the uniqueness of F. Now, let $F' : X \to Y$ be another quadratic–additive mapping satisfying (2.5). Then

$$F'(x) - J_m F'(x) = \sum_{i=0}^{m-1} \left(\frac{4^{\tau_{-s,i}}}{2^{n-1}} DF'(2^{\tau_{s,i}}x, 2^{\tau_{s,i}}x, 0, \ldots, 0)s + \frac{4^{\tau_{-s,i}}}{2^{n-1}} DF'(-2^{\tau_{s,i}}x, -2^{\tau_{s,i}}x, 0, \ldots, 0)s + \frac{2^{\tau_{-t,i}}}{2^{n-1}} DF'(2^{\tau_{t,i}}x, 2^{\tau_{t,i}}x, 0, \ldots, 0)t - \frac{2^{\tau_{-t,i}}}{2^{n-1}} DF'(-2^{\tau_{t,i}}x, -2^{\tau_{t,i}}x, 0, \ldots, 0)t \right) = 0$$

for all $m \in \mathbb{N}$.

From this and (2.5), we obtain

$$\begin{aligned} \|F(x) - F'(x)\| &\leqslant \|J_m F(x) - J_m F'(x)\| \\ &\leqslant \frac{4^{-sm}}{2} \left(\|(f - F)(2^{sm}x)\| + \|(f - F')(2^{sm}x)\| + \|(f - F)(-2^{sm}x)\| + \|(f - F')(-2^{sm}x)\|\right) \\ &\quad + \frac{2^{-tm}}{2} \left(\|(f - F)(2^{tm}x)\| + \|(f - F')(2^{tm}x)\| + \|(f - F)(-2^{tm}x)\| + \|(f - F')(-2^{tm}x)\|\right) \end{aligned} \tag{2.7}$$

for all $x \in X$ and all $m \in \mathbb{N}$.

It follows from (2.5) and (2.7) that

$$\|F(x) - F'(x)\| \leqslant \sum_{i=0}^{\infty}\left(4^{-sm}\left(\Phi_i(2^{sm}x) + \Phi_i(-2^{sm}x)\right) + 2^{-tm}\left(\Phi_i(2^{tm}x) + \Phi_i(-2^{tm}x)\right)\right)$$

for all $x \in X$ and all $m \in \mathbb{N}$. Taking the limit as $m \to \infty$ in the above inequality and using the equality $F(0) = 0 = F'(0)$, we can conclude that $F(x) = F'(x)$ for all $x \in X$. This proves the uniqueness of F. $\square$

By the similar method used in the proof of Theorems 2.2 and 2.3, we prove the following corollary.

Corollary 2.4. *Let $p \notin \{1,2\}$ be a nonnegative real number. Suppose $f : X \to Y$ is a mapping such that*

$$\|Df(x_1, x_2, \ldots, x_n)\| \leqslant \theta\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right) \tag{2.8}$$

for all $x_1, x_2, \ldots, x_n \in X$ and for some constant $\theta \geqslant 0$, where $f(0) = 0$ is assumed provided $p = 0$. Then there exists a unique quadratic–additive mapping F such that

$$\|f(x) - F(x)\| \leqslant \left(\frac{1}{|2^p - 4|} + \frac{1}{|2^p - 2|}\right)\frac{\theta\|x\|^p}{n-1} \tag{2.9}$$

for all $x \in X$.

Proof. If $p < 1$ or $p > 2$, then this corollary follows from Theorem 2.3. In view of Theorem 2.2, if $1 < p < 2$, then there exists a mapping F satisfying $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X$ as well as (2.9) for all $x \in X$ with $F(0) = 0$.

Now, let $F' : X \to Y$ be another mapping satisfying (2.9) with $F'(0) = 0$. Using Lemma 2.1, (2.7) and (2.9), we obtain

$$\|F(x) - F'(x)\| \leqslant \left(\frac{2}{|2^p - 4|} + \frac{2}{|2^p - 2|}\right)\left(\left(2^{p-2}\right)^m + \left(2^{1-p}\right)^m\right)\frac{\theta\|x\|^p}{n-1}$$

for all $x \in X$ and all $m \in \mathbb{N}$. Taking the limit as $m \to \infty$ in the above inequality and using the equality $F(0) = 0 = F'(0)$, we can conclude that $F(x) = F'(x)$ for all $x \in X$, which proves the uniqueness of F. $\square$

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